The University of Nottingham

ANALYSIS OF SINGLE-DEGREE-OF-FREEDOM SYSTEMS
MODULE: MMME2046 DYNAMICS & CONTROL

Initially, we will look at structures that can be modelled with one degree of freedom. This is the simplest model for a vibrating structure. The model normally takes the form of a rigid body restrained by one or more massless springs. It works well for structures with one resonance or where one resonance dominates the vibration behaviour. It gives good insight into vibration behaviour and is often used as a first approximation for more complicated structures.

We will look at a number of different examples where single-degree-of-freedom dynamic models are appropriate, but the general approach to the analysis will be the same in all cases and will involve the following steps.

1) Convert the ***physical structure*** into a ***dynamic mass-spring model***.
2) Draw a ***free body diagram***. Displacing the body from its equilibrium position will create forces in the restraining springs that will try to return the body to equilibrium. The free body diagram can be thought of as a "snapshot" of the state of the system when it has moved away from equilibrium by a chosen amount. Drawing the free body diagram is the key step in the analysis and should be tackled systematically. There are three stages.
    - (i) Start with the system in equilibrium and draw it as a free body. To create the "free" body, draw it without any of the restraining springs. The forces exerted by the springs will be added in stage (iii).
    - (ii) Select a motion coordinate to describe how the system will deflect from its equilibrium position and mark it on the diagram.
    - (iii) Apply a positive deflection in the chosen motion coordinate, identify the forces (and/or moments) that result and draw them on the diagram. It is critical that the positive directions of the forces due to a positive deflection are shown correctly.
3) Apply the appropriate form of Newton's 2nd Law of motion to give the ***equation of motion*** for the system.

The following pages show how steps (1), (2) and (3) can be applied to a number of examples. Others will crop up later in the module. The main source of mistakes lies in applying the steps incorrectly (some students even attempt to miss some steps out!) and it is vital that you **adopt a systematic approach using the three steps when setting up solutions**. See how the problems on the next pages are developed and follow the same pattern yourself.

- **At this stage, we will concentrate on finding the *NATURAL FREQUENCY* of the systems. This is the frequency at which a system will vibrate when displaced from equilibrium and then released. Later, we will consider the effects of external excitation and damping, but these are omitted for the moment.**

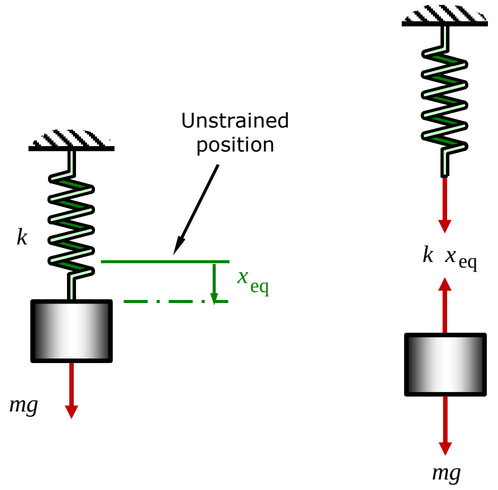
When we look at the effects of excitation, we will find that for most engineering structures there is a maximum response if the excitation frequency coincides with the natural frequency (which is why we need to know what its value is). This effect is called ***resonance*** and the term ***resonant frequency*** is often used instead of natural frequency (although, strictly speaking, the two are different as we shall see later).

Example 1 Simple Mass-spring System

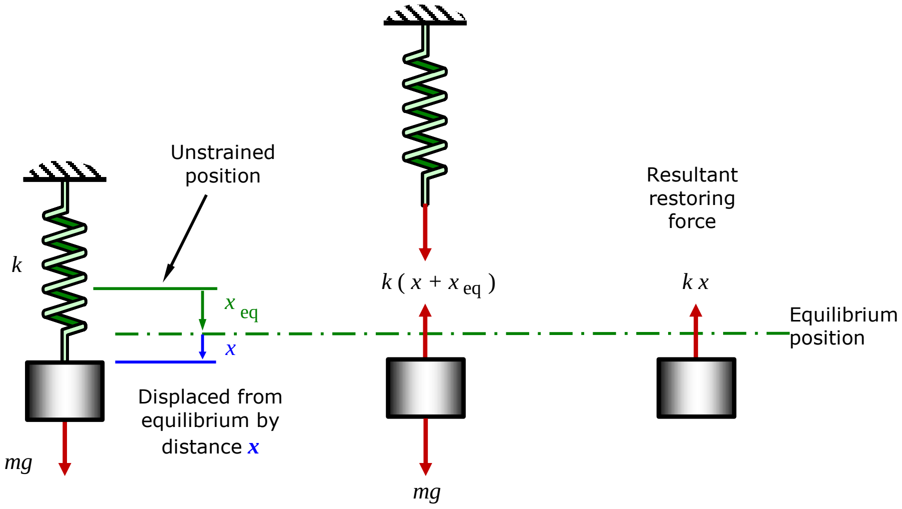
If a mass m (kg) is suspended from a spring of stiffness k (N/m), it will move down under the effect of gravity and stretch the spring by a distance x_{eq} before reaching its static equilibrium position.

Once in equilibrium, the resultant force on the mass is zero and hence:

$$mg = kx_{eq}$$

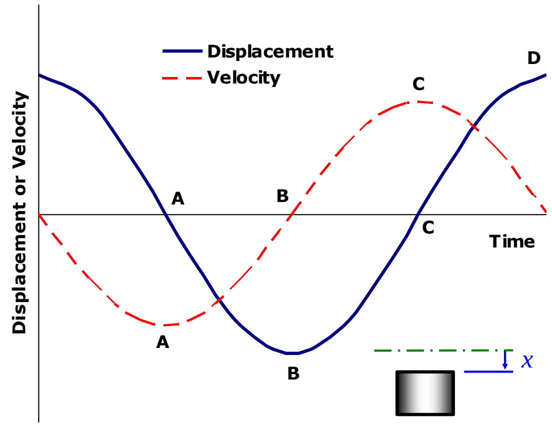


What then happens if the mass is given a *further* downward displacement, x, away from equilibrium and then released?



The displacement x produces an additional force in the spring of kx, as shown in the central diagram above. Since we know from the static equilibrium case that $mg = kx_{eq}$, the resultant force on the mass is kx, as shown in the right-hand diagram.

Remembering that the displacement and the force are both vectors, we see that a positive downward displacement from equilibrium produces a positive upward resultant force on the mass. This acts to return it to its equilibrium position. At any instant when x was negative (meaning that the mass was *above* its equilibrium position), the "upward" resultant force, kx, would also be negative, telling us that the force was actually acting downwards at this instant. Since the force always acts to return the mass to equilibrium, the term ***restoring force*** is often used.

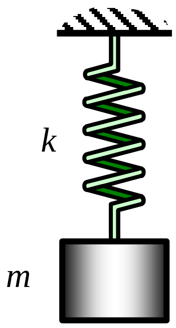


So, what happens when the mass is displaced downwards (x positive) away from equilibrium and then released? Initially, there will be an upward resultant force on the mass, so it will accelerate upwards. Since downward movement has been chosen as positive, upward velocity is negative. When the mass reaches its equilibrium position (point A in the figure on the previous page), x is zero and the resultant force is zero as well. At this point, the velocity has its maximum upward (i.e., negative) value. This upward velocity carries the mass past this point and it moves above the equilibrium position. x now becomes negative and the resultant force then acts downwards and slows the mass down. At point B, the mass reaches its maximum upward (i.e., negative) displacement and is instantaneously at rest. Since x is still negative, the resultant force continues to act downwards. The mass passes back through the equilibrium position (point C), when it has its maximum downward (i.e., positive) velocity. x becomes positive once more and the upward resultant force slows the mass down until it reaches its original starting position (point D).

We see that the resultant force on the mass depends only on the displacement measured from the equilibrium position. Here, and in all other problems, the static forces in springs exactly balance any gravitational forces on the mass under equilibrium conditions. Because they always cancel each other, we ignore them and start at the equilibrium position and consider displacements away from that position.

Note: We assume that displacements are small. In particular, we assume that the stiffness of the spring is constant. In real systems, stiffness may vary with displacement.

Physical System



STEP 1: Dynamic mass-spring model

STEP 2: Free Body Diagram

(i) Remove the spring to leave the mass by itself.

(ii) Mark the chosen positive direction for displacement.

(iii) Give the mass a positive displacement, write down the expression for the force and add it to the diagram to show its positive direction.

STEP 3: Equation of motion

This can be re-arranged to give an equation in the form of a second-order ordinary differential equation.

Any sinusoidal function of appropriate periodicity satisfies this equation, but from the earlier description of what happens to the mass when it is displaced downwards and then released, a suitable mathematical form would be $x(t) = X \cos\omega t$. This describes a sinusoidally varying displacement at frequency ω, with maximum deflections of X above and below the equilibrium position. The frequency of the vibration is called the ***natural frequency*** and is the characteristic quantity we are trying to find. Sinusoidal displacement like this is often referred to as ***simple harmonic motion*** and many of you will have come across this before.

Substitute for $x(t)$ into the equation of motion.

The natural frequency for this system is therefore given by $\sqrt{\frac{k}{m}}$. The symbol, ω_n, is normally used for the natural frequency. When used in the equation of motion, ω must have the units of **rad/s** to make the equation consistent. However, the value would normally be quoted (in a report, for example) using the units of **Hz** (Hertz or cycles/s). The two are linked by the equation:

$$f_n\ [\mathrm{Hz}] \quad = \quad \frac{\omega_n}{2\pi}\,[\mathrm{rad/s}]$$

This is an example of where the SI system of units can catch you out. The rule is: **when substituting frequency values into formulae, use units of rad/s**. When quoting frequency values in answers or reports, use units of Hz.

You will find that other systems have different equations of motion, but all will have the same form, namely:

$$M\ddot{z} \quad + \quad Kz \quad = \quad 0$$

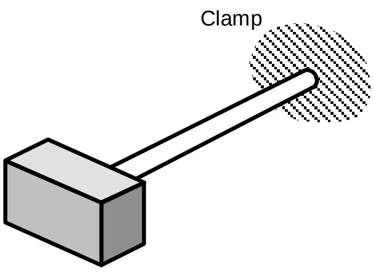
where z is the chosen motion coordinate. For some systems, the expressions for the coefficients M and K can be quite complicated. Following the above analysis, we would find that the natural frequency would be given by

$$\omega_n \quad = \quad \sqrt{\frac{K}{M}} \quad [\mathrm{rad/s}]$$

Hence, as soon as you've derived the equation of motion and obtained the coefficients of the displacement and acceleration, you can immediately write down the expression for the natural frequency of a system. **Indeed, this is the way you should always do it.**

Example 2 Vertical vibration of a block on a flexible cantilever beam

Physical System



The block will be treated as a rigid mass and the supporting beam as a massless spring. The **bending stiffness** of the beam is given by $k_B = \dfrac{3EI}{L^3}$ (this is on the Formula Sheet). See the note at the bottom of page 5 relating to the symbol ***I*** in this formula.

STEP 1: Dynamic Model

Example 3 Torsional vibration of a block on a cantilever beam

For torsional motion, the support beam can be modelled as a spring with **torsional stiffness** $k_T = \dfrac{GJ}{L}$ (with units of Nm/rad). The block can again be modelled as a rigid body, but with a moment of inertia I about the beam axis[1], which we will assume to be a fixed axis.

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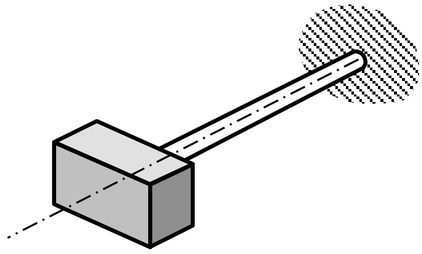
1 The symbol ***I*** is commonly used by Engineers for two different terms. In problems involving beam bending, such as example 2, ***I*** is used for the "***second moment of area***". This has units of [length^4]. In example 3, ***I*** is used for the "***moment of inertia***", which has units of [mass x length^2]. **The two terms are completely different** and you should take care not to confuse them.

STEP 1: Dynamic Model



STEP 2: Free Body Diagram

STEP 3: Equation of Motion

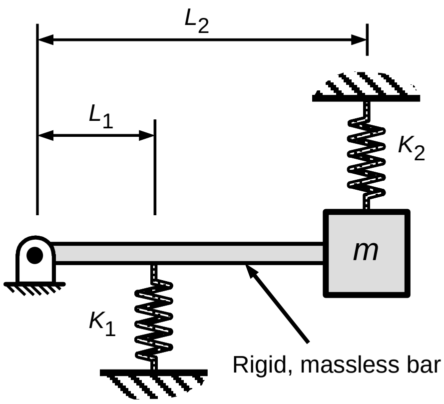
The natural frequency for this system is

Example 4 Rocker System

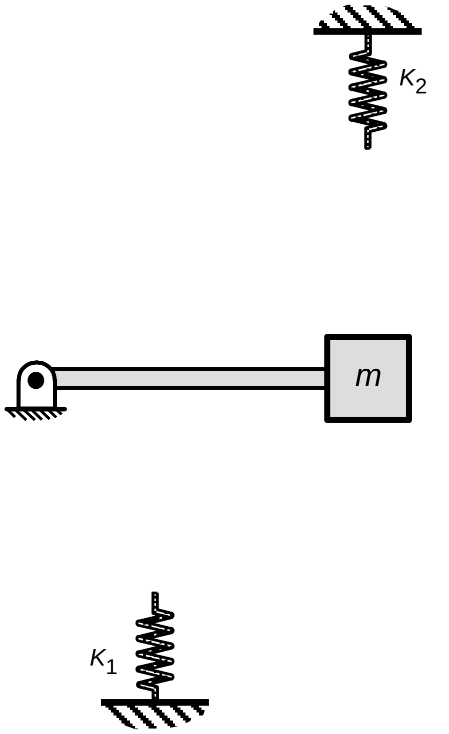
The previous examples are very simple systems. Example 4 is more typical of what you will meet in this module. It consists of a rigid, massless bar with a fixed pivot at one end and a large mass attached at the other. The rocking motion about the pivot is restrained by two springs, one attached to the mass and the other that is connected to the bar, part way along its length.

Note: We will use the angular displacement of the bar about the fixed pivot as the motion coordinate and assume that this displacement is ***small***. This means that $\cos\theta = 1$ and $\sin\theta = \tan\theta = \theta$. Taken together with the earlier assumption that small displacements mean that we can assume that stiffness values are constant, the model will give an equation of motion that is a second order ordinary differential equation with constant coefficients.

STEP 1: Dynamic Model



STEP 2: Free body diagram[2]



STEP 3: Equation of Motion

We can write down the expression for the natural frequency from the coefficients of $\ddot{\theta}$ and θ.

2 If you're worried that gravity is again ignored, remember that as the bar moves down under the effect of gravity, K_1 will compress and K_2 will extend. Once in equilibrium, the moment that the force *mg* exerts about the pivot will be exactly balanced by the moments from the forces in the two springs. All of these forces are still there in the vibrating system, but since they cancel out, we don't need to include them in the equation.

Advice for Tackling Vibration Problems

As systems become more complicated, the scope for making mistakes increases. **You must adopt a systematic approach to setting out problems.** This approach is summarised by the three steps given on page 1 and will be used for all of the examples presented in the lectures. You should follow the same three steps when you tackle problems.

Fortunately, when you reach the end of Step 3 and you have the equation of motion, it's possible to tell immediately if it's wrong!

- **For all real vibrating systems, every term in the expressions for the coefficients of both the displacement and acceleration MUST be positive.**

- **If you find a negative sign anywhere, you can be certain that you have made a mistake somewhere.**

The most common mistakes are:

1. One or more forces on the free body diagram may have been drawn in the wrong direction.
2. Forces or moments may not have been resolved in the same direction as the motion coordinate.

Because we can make this check for errors, **you should derive the equation of motion using symbols** and only substitute the numerical values at that stage.[3]

Unfortunately, there is no way of knowing that the equation is correct, but being able to tell if it's wrong is a good start!

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3 In case it's not obvious, someone could work out the coefficient for θ in Example 4 to be $K_1 L_1^2 - K_2 L_2^2$ and, depending on the stiffness and length values given, could still get a positive numerical value for the coefficient. The value would, of course, be wrong since the expression itself is wrong. Working with symbols up to this point makes the presence of the negative sign apparent.